拉格朗日拟序结构 LCS

Finite Time Lyapunov Exponents FTLE

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图 1: A photograph of Vincent van Gogh's The Starry Night (1889), which currently hangs in the Museum of Modern Art in New York.

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拉格朗日拟序结构 LCS

Lagrangian Coherent Structure(Haller 等, 2000)： the most repelling (**排斥**), attracting (**吸引**), … material surfaces (**物质界面**) …

拉格朗日拟序结构 LCS

Lagrangian Coherent Structure(Haller 等, 2000)： the most repelling (排斥), attracting (吸引), ... material surfaces (物质界面) ...

图 2: Kármán vortex street behind a circular cylinder at $Re = 140$. Integrated streaklines are shown by electrolytic precipitation of a white colloidal smoke, illuminated by a sheet of light.

Book link

LCS 与 FTLE

拉格朗日拟序结构 LCS

Lagrangian Coherent Structure(Haller 等, 2000)： the most repelling (排斥), attracting (吸引), … material surfaces (物质界面) …

有限时间李亚普诺夫指数 FTLE

Finite-Time Lyapunov Exponents (FTLE) can be used to find separatrices in time-dependent systems.

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拉格朗日视角的动力系统描述

微分形式：

Passive particle

$$\frac{d}{dt}x = u(x,t). \qquad (1)$$

Unsteady fluid flow
(vector field)

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Unsteady fluid flow
(vector field)

积分形式：

$$x(t) = \underbrace{x_0 + \int_{t_0}^{t} u(x(\tau),\tau)d\tau}_{F_{t_0}^{t}(x_0)\text{ flow map}}\,. \qquad (2)$$

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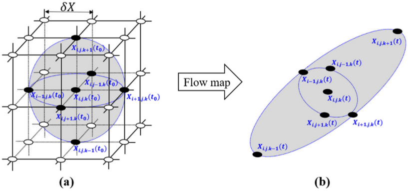

图 4: (a) filled-circle points and empty-circle points are initial conditions of all trajectories at initial time t_0; (b) filled-circle points are final positions of trajectories at time t for the flow map derivative at position $x_{i,j,k}(t_0)$. (Li 等, 2022)

稳定性与不稳定性

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y + x^2. \end{cases} \qquad (3)$$

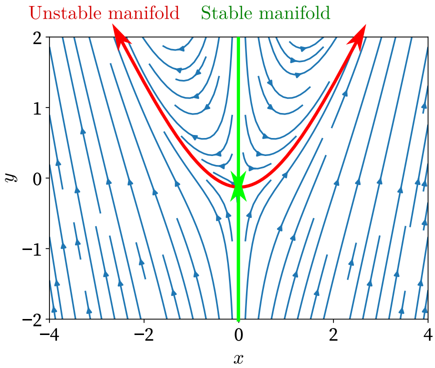

图 5: 式 (3) 的相图

稳定性与不稳定性

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y + x^2. \end{cases} \tag{3}$$

- unstable manifold (不稳定流形) $\Longrightarrow$ attract (吸引)
- stable manifold (稳定流形) $\Longrightarrow$ stretch (延伸) /repell (排斥)

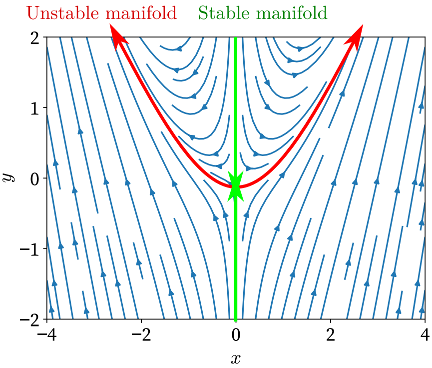

图 5: 式 (3) 的相图

吸引结构与排斥结构

- **不稳定流形** (unstable manifold) $\Longrightarrow$ **吸引结构** (attracting structures)
- **稳定流形** (stable manifold) $\Longrightarrow$ **排斥结构** (repelling structures)

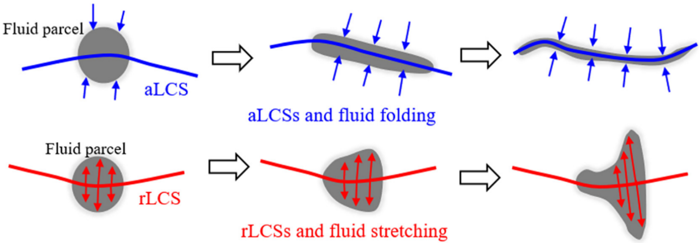

图 6: The role of attracting (aLCS) and repelling (rLCS) LCSs in fluid deformation. (Li 等, 2022)

吸引结构与排斥结构

- 不稳定流形 (unstable manifold) $\Longrightarrow$ 吸引结构 (attracting structures)
- 稳定流形 (stable manifold) $\Longrightarrow$ 排斥结构 (repelling structures)

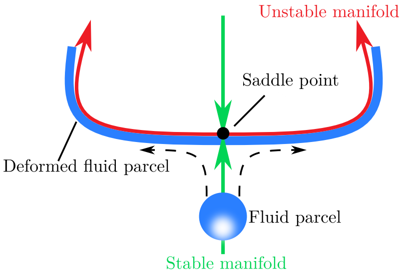

图 7: A fluid parcel approaching the saddle point astride one material line (the repelling stable manifold) eventually becomes drawn out and away from the saddle point along the orthogonal material line (the attracting unstable manifold). (Peacock 等, 2013)

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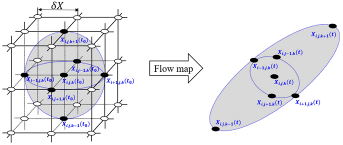
计算流程

- 差值重建 flow map $F_{t_0}^t$



(a) (b)

- 计算 flow map Jacobian

$$\nabla F_{t_0}^t \approx \begin{bmatrix} \frac{\Delta x(t)}{\Delta x(t_0)} & \frac{\Delta x(t)}{\Delta y(t_0)} \\ \frac{\Delta y(t)}{\Delta x(t_0)} & \frac{\Delta y(t)}{\Delta y(t_0)} \end{bmatrix}.$$

- 计算 FTLE

$$\sigma(F_{t_0}^t; x_0) = \frac{1}{|t - t_0|} \log\left(\sqrt{\lambda_{\max}\left[(\nabla F_{t_0}^t)^T \nabla F_{t_0}^t\right]}\right).$$

差值重建 flow map

欧拉视角到拉格朗日视角的转化

$$\frac{d}{dt}\boxed{x} = u(x,t) \implies x_{i,j}(t+\delta t) \approx x_{i,j}(t) + \boxed{v_{i,j}(t)} \cdot \delta t.$$

Passive particle

flow velocity field

差值重建 flow map

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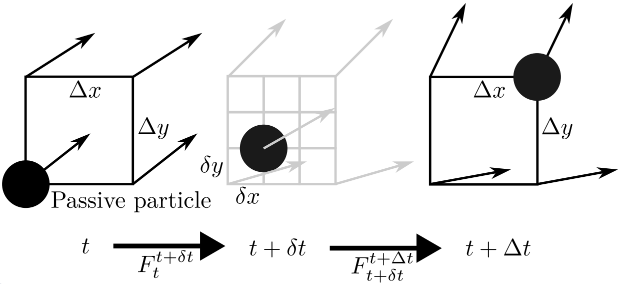
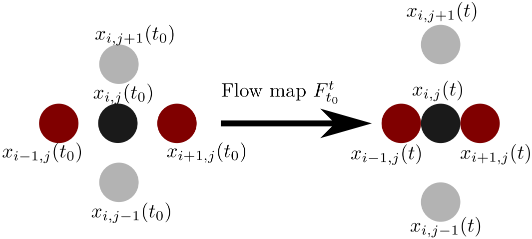
$$\frac{d}{dt}\boxed{x} = u(x,t) \implies x_{i,j}(t+\delta t) \approx x_{i,j}(t) + \boxed{v_{i,j}(t)} \cdot \delta t.$$

Passive particle

flow velocity field



计算 flow map Jacobian



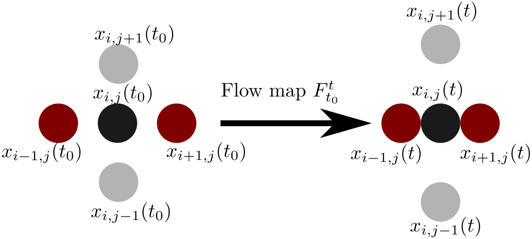
$$\nabla F_{t_0}^t \approx \begin{bmatrix} \frac{\Delta x(t)}{\Delta x(t_0)} & \frac{\Delta x(t)}{\Delta y(t_0)} \\ \frac{\Delta y(t)}{\Delta x(t_0)} & \frac{\Delta y(t)}{\Delta y(t_0)} \end{bmatrix} = \begin{bmatrix} \frac{x_{i+1,j}(t)-x_{i-1,j}(t)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{x_{i+1,j}(t)-x_{i-1,j}(t)}{x_{i,j+1}(t_0)-x_{i,j-1}(t_0)} \\ \frac{x_{i,j+1}(t)-x_{i,j-1}(t)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{x_{i,j+1}(t)-x_{i,j-1}(t)}{x_{i,j+1}(t_0)-x_{i,j-1}(t_0)} \end{bmatrix}. \tag{4}$$

计算 FTLE

FTLE: 衡量有限时间内的最大拉伸

$$\sigma(F_{t_0}^{t}; x_0) = \frac{1}{|t - t_0|} \log \left(\sqrt{\lambda_{\max} \left[(\nabla F_{t_0}^{t})^{T} \nabla F_{t_0}^{t} \right]} \right). \tag{5}$$

max eigenvalue of
right Cauchy-Green strain tensor



前向积分与后向积分

flow map $F_{t_0}^t$:

$$x(t) = \underbrace{x_0 + \int_{t_0}^{t} u(x(\tau), \tau) d\tau}_{F_{t_0}^t(x_0) \text{ flow map}} .$$

- **前向** (forward) **积分** $\Longrightarrow$ **排斥结构** (repelling structures): $t > t_0$

前向积分与后向积分

flow map $F_{t_0}^{t}$:

$$x(t) = \underbrace{x_0 + \int_{t_0}^{t} u(x(\tau),\tau)d\tau}_{F_{t_0}^{t}(x_0)\ \text{flow map}}.$$

- 前向 (forward) 积分 $\Longrightarrow$ 排斥结构 (repelling structures): $t > t_0$
- 后向 (backward) 积分 $\Longrightarrow$ 吸引结构 (attracting structures): $t < t_0$

积分时间长度

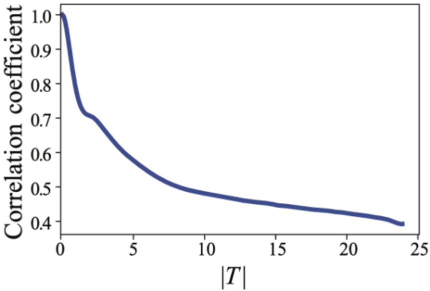


图 8: Pearson correlation coefficient between the attraction rate field and the benchmark backward-time FTLE field as a function of integration time, $|T|$, in hours. (Nolan 等, 2020)

例子

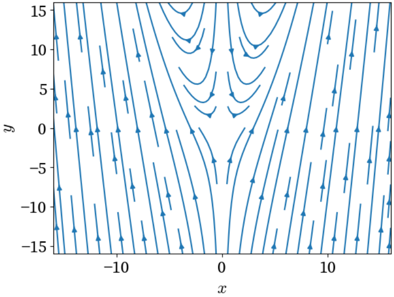


图 9: 式3相图

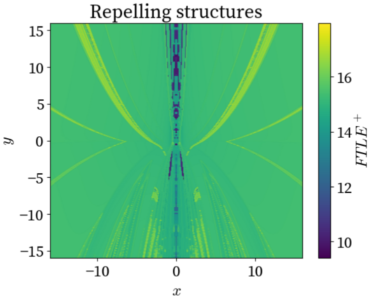


图 10: 前向积分 $|t - t_0| = 6$, 排斥结构

例子

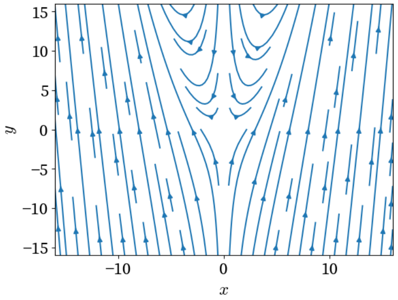


图 11: 式3相图

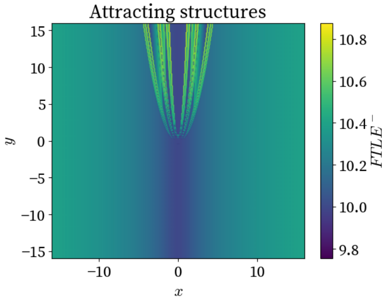


图 12: 后向积分 $|t - t_0| = 6$, 吸引结构

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总结

- LCS 定义(Haller 等, 2000)： the most repelling, attracting material surfaces.
- Finite-Time Lyapunov Exponents (FTLE) can be used to find separatrices in time-dependent systems.

参考文献 I

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