K41 理论

Big whorls, little whorls - 2024 年 4 月 15 日

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湍流唯象学

Big whorls, little whorls

Big whorls have little whorls

Which feed on their velocity,

And little whorls have lesser whorls

And so on to viscosity.

图 1 **Studies of water (RCIN 912661).**

湍流尺度

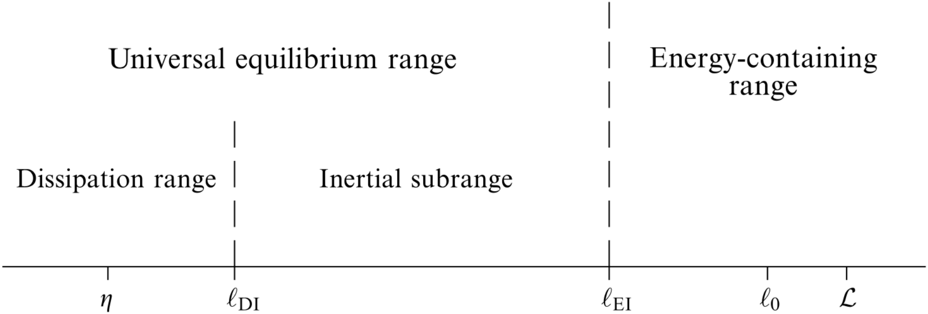


图 2 **湍流涡体尺度分布**[1].

- $\mathcal{L}$: 高雷诺数充分发展湍流的特征长度
- ℓ_0: 最大尺度涡的特征长度($\ell_0 \sim \mathcal{L}$)
- η: 最小尺度涡的特征长度(Kolmogorov 尺度)

能量串级

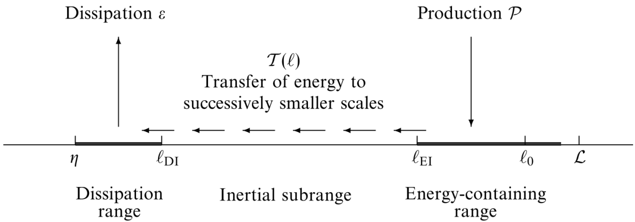
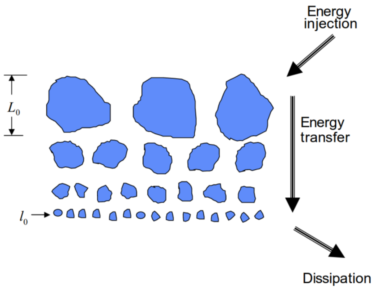


图 3　**高雷诺数情况下的（正向）能量串级示意图**[1]**.**



K41a/c 文献

K41a/c 原文

The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers†

BY A. N. KOLMOGOROV

图 5　**K41a[2].**

Dissipation of energy in the locally isotropic turbulence†

BY A. N. KOLMOGOROV

图 6　**K41c[3].**

流速结构函数的 2/3 律

实验结果表明，二阶流速结构函数满足 2/3 律:

$$S_2(r) = \langle [u_1(x+r) - u_1(x)]^2 \rangle \sim r^{\frac{2}{3}}. \tag{1}$$

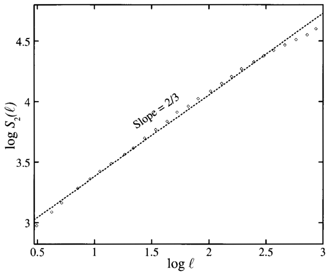


图 7 **双对数坐标下二阶流速结构函数**[4]**.**

- p_{th} order velocity strcture function:

$$S_p(r) = \langle \left[\left(\vec{u}\left(\vec{x} + r\vec{l0}\right) - \vec{u}(\vec{x}) \right) \cdot \vec{l0} \right]^p \rangle. \tag{2}$$

K41a: Universal similarity hypothesises

Hypothesis 1: first similarity.

In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions ($\ell < \ell_{EI}$) have a **universal form** that is **uniquely determined by** ν **and** $\bar{\varepsilon}$. (Kolmogorov, 1941a)

由 H1 并结合量纲分析得到 Kolmogorov 尺度: $\eta = \left(\frac{\nu^3}{\bar{\varepsilon}}\right)^{\frac{1}{4}}$.

Hypothesis 2: second similarity.

In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale ℓ in the range $\ell_0 \gg \ell \gg \eta$ have a **universal form** that is **uniquely determined by** $\bar{\varepsilon}$, independent of ν.

由 H2 并结合量纲分析得到二阶流速结构函数的 2/3 律：

$$S_2(r) = C\bar{\varepsilon}^{\frac{2}{3}}r^{\frac{2}{3}}. \tag{3}$$

K41a 困境: the lack of universality

- 公式 3 中的常数C不是一个普适的常数，在不同湍流工况中不一致。

朗道的质疑[5,6]: ***The result of the averaging therefore cannot be universal.***

考虑N次独立的湍流测量结果，得到总体平均(ensemble average)的二阶流速结构函数$S_2(r)$和湍动能耗散率$\bar{\varepsilon}$为:

$$\begin{cases} S_2(r) = \frac{1}{N}\sum_i S_2^i(r) = \frac{1}{N}\sum_i C\varepsilon_i^{\frac{2}{3}} r^{\frac{2}{3}}, \\ \bar{\varepsilon} = \frac{1}{N}\sum_i \varepsilon_i. \end{cases} \tag{4}$$

公式 3 成立意味着

$$\frac{1}{N}\sum_i \varepsilon_i^{\frac{2}{3}} = \left(\frac{1}{N}\sum_i \varepsilon_i\right)^{\frac{2}{3}}. \text{等式不成立} \tag{5}$$

K41c: 三阶流速结构函数的 4/5 律

Kolmogorov[3] 通过 Karman-Howarth-Monin 公式[7] 从理论上推导了:

$$S_3(r) = \langle [u_1(x+r) - u_1(x)]^3 \rangle = \frac{4}{5}\bar{\varepsilon} r. \tag{6}$$

Kolmogorov 假设流速差$\Delta u_1(r) = u_1(x+r) - u_1(x)$的概率分布偏度 (skewness)$S$为常数:

$$S = \frac{S_3(r)}{S_2(r)^{\frac{3}{2}}} = \text{const.} \tag{7}$$

因此从理论上阐释了二阶流速结构函数的 2/3 律:

$$S_2(r) = \left(-\frac{4}{5S}\right)^{\frac{2}{3}} (\bar{\varepsilon} r)^{\frac{2}{3}} \sim r^{\frac{2}{3}}. \tag{8}$$

he assumes that the skewness is 'constant' (independent of scale) rather than 'universal' (independent of the flow).[4]

K41 理论的推论

Corollary 1: 湍流惯性区功率谱密度-5/3 律.

$$E(k) \sim \bar{\varepsilon}^{\frac{2}{3}} k^{-\frac{5}{3}}. \tag{9}$$

二阶流速结构函数$S_2(r)$可表示为相关函数$R_{11}(r) = \langle u_1(x+r) u_1(x) \rangle$的形式:

$$S_2(r) = 2R_{11}(0) - R_{11}(r) - R_{11}(-r) = 2R_{11}(0) - 2R_{11}(r). \tag{10}$$

由于相关函数$R_{11}(r)$与功率谱密度$E(k)$为一对 Fourier 变换对（维纳-辛钦定理），因此功率谱密度可表示为二阶流速结构函数的 Fourier 变换形式:

$$S_2(r) = 2\int_{-\infty}^{\infty} \left(1 - e^{ik\cdot r}\right) E(k) dk. \tag{11}$$

由公式 11，从二阶流速结构函数的 2/3 律导出功率谱密度的-5/3 律.

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[7] KARMAN T de, HOWARTH L. On the Statistical Theory of Isotropic Turbulence[J]. Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences, 1938, 164(917): 192-215.